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SOME REMARKS ABOUT EMBEDDINGS OF l_1^k IN FINITE-DIMENSIONAL SPACES

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ABSTRACT

The span X_n of functions $x_i(t) = \pm 1$, $i = 1, \cdots, n$, on a set T in the supremum norm is considered. It is proved, for example, that X_n contains an isometric copy of l_1^k for $k \geqq cM_n^2/n \log n$ where M_n is the Rademacher average of $\{x_i\}_1^n$. This generalizes a result of Pisier for characters. The proof uses a new combinatorial tool.

1. We use the standard notations of Banach theory which may be found, for example, in [5].

Let G be a compact Abelian group with dual group Γ. For a set A, $|A|$ will denote the cardinality of A.

The starting point of this paper is the following remarkable result of G. Pisier [11].

1.1. THEOREM (G. Pisier). *Let $A \subset \Gamma$ be a finite set of characters G. Define the number*

$$M = \operatorname*{Average}_{\varepsilon_\gamma = \pm 1} \left\| \sum_{\gamma \in A} \varepsilon_\gamma \gamma(t) \right\|_{C(G)}.$$

There exists a subset $B \subset A$, $|\beta| \geqq \alpha M^2/|A|$ (where α is an absolute constant), such that B is a Sidon set with some absolute constant d (this means that $\{\gamma\}_{\gamma \in B}$ in C-norm is d-equivalent to the natural basis in $l_1^{|B|}$).

The original proof of Theorem 1.1 is long and complicated. We give below a simple and short proof of the theorem for the case of the real characters. But the main purpose of section 2 of this paper is some generalizations (see Theorems 2.1 and 2.3). For example, we completely ignore the group structure of G. The new point in our proof is the use of the following combinatorial lemma.

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1.2. Let E_2^n be an n-dimensional space over the field of two elements $\{-1,+1\}$. So $E_2^n = \{\vec{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_n) : \varepsilon_i = \pm 1\}$.

LEMMA. *Let* $S \subset E_2^n$ *and* $|S| > \sum_{j=0}^{k-1} \binom{n}{j}$. *Then there exists a subset* $A \subset [1, \cdots, n]$, $|A| = k$, *such that the restriction* $S|_A$ *includes all* 2^k *different vectors on coordinates from* A.

This lemma was formulated in three different manners (and was proved in three different ways) in [12], [13] and [3]. In the paper [3] it was formulated in a more general form which may be useful for Banach space theory.

All the results on finite-dimensional spaces to be discussed here are asymptotic (for high dimensions). This is the reason why, whenever we speak of topological or geometrical properties of an n-dimensional space E, we actually mean some family $X = \{E_n, \dim E_n \to \infty \ (n \to \infty)\}$ of spaces E_n such that all spaces of the family with sufficiently large dimensions possess (and uniformly in n) the indicated properties. We usually do not personalize absolute constants and we may use the same letter c for different numbers.

2. In this section we always assume that $x_i(t)$ are real-valued functions on a set T and $|x_i(t)| = 1$; on $X_n = \mathrm{span}\{x_i(t)\}_1^n$, $n > 1$, we consider the supremum norm on T. So $\|y\| = \sup\{|y(t)| : t \in T\}$ and, for $K \subseteq T$,

$$\|y\|_{C(K)} = \sup\{|y(t)|, t \in K\}.$$

2.1. THEOREM. *Let* $M_n = \mathrm{Average}_{\varepsilon_i = \pm 1} \|\sum_1^n \varepsilon_i x_i(t)\|$. *Then there exists a set* $A \subset [1, \cdots, n]$ *such that* $\mathrm{span}\{x_i\}_{i \in A}$ *is isometric to* $l_1^{|A|}$ *and* $|A| \geqq [M_n^2/10n \log n] - 2$. *Moreover* $\{x_i\}_{i \in A}$ *is isometrically equivalent to the natural basis of* $l_1^{|A|}$.

PROOF. For every vector of signs $\vec{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_n) \in E_2^n$ let $T_{\vec{\varepsilon}} = \{t \in T;\ \mathrm{sign}\, x_i(t) = \varepsilon_i\}$. Let $S = \{\vec{\varepsilon}\,; T_{\vec{\varepsilon}} \neq \emptyset\}$ and $k = [M_n^2/10n \log n] - 2$.

Case 1. There exists an $A \subset [1, \cdots, n]$, $|A| = k$ and $|S|_A| = 2^k$ (i.e. all possible k-vectors of signs are in $S|_A$). In this case, for any set of real numbers $\{a_i\}_{i \in A}$ there exists an $\vec{\varepsilon} \in S$ and $t_{\vec{\varepsilon}} \in T_{\vec{\varepsilon}}$ such that $a_i x_i(t_{\vec{\varepsilon}}) = |a_i|$ for all $i \in A$. Then

$$\left\| \sum_{i \in A} a_i x_i(t) \right\| \geqq \sum_{i \in A} a_i x_i(t_{\vec{\varepsilon}}) = \sum_{i \in A} |a_i|$$

and span $\{x_i\}_{i \in A}$ is isometric to $l_1^{|A|}$.

Case 2. There exists no A as in Case 1. Then by Lemma 1.2, $|S|$ is quite small, precisely,

$$|S| \leqq \sum_{i=0}^{k-1} \binom{n}{i} < kn^k \qquad (\text{since } k < n/2).$$

Define, for $K \leqq E_2^n$, $\mu(K) = |K| 2^{-n}$ and, for $d > 0$ and $\vec{\varepsilon} \in S$,

$$E_{\vec{\varepsilon}}(d) = \left\{ \vec{\delta} \in E_2^n : \left\| \sum_{i=1}^n \delta_i x_i \right\|_{C(T_{\vec{\varepsilon}})} \leqq d \right\}.$$

Then,

$$\mu(E_{\vec{\varepsilon}}(d)) = \mu\left\{ \vec{\delta} : \left| \sum_{i=1}^n \delta_i \varepsilon_i \right| \leqq d \right\} = 2^{-n} \sum_{i=(n-d)/2}^{(n+d)/2} \binom{n}{i} \geqq 1 - 2\exp\left(\frac{-d^2}{4n}\right).$$

It follows that

$$\mu\left(\bigcap_{\vec{\varepsilon} \in S} E_{\vec{\varepsilon}}(d) \right) \geqq 1 - 2|S| \exp\left(\frac{-d^2}{4n}\right) \geqq 1 - 2kn^k \exp\left(\frac{-d^2}{4n}\right).$$

Denote by Σ' the sum over $\vec{\delta} \notin \bigcap_{\vec{\varepsilon} \in S} E_{\vec{\varepsilon}}(d)$ and by Σ'' the sum over $\vec{\delta} \in \bigcap_{\vec{\varepsilon} \in S} E_{\vec{\varepsilon}}(d)$; then

$$M_n = \operatorname*{Ave}_{\delta_i = \pm 1} \left\| \sum_{i=1}^n \delta_i x_i \right\| = 2^{-n} \sum{}' \max_{\vec{\varepsilon} \in S} \left\| \sum_1^n \delta_i x_i \right\|_{C(T_{\vec{\varepsilon}})} + 2^{-n} \sum{}'' \max_{\vec{\varepsilon} \in S} \left\| \sum_1^n \delta_i x_i \right\|_{C(T_{\vec{\varepsilon}})}$$

$$\leqq 2kn^{k+1} \exp(-d^2/4n) + d.$$

For $d = \frac{2}{3} M_n$ we get $M_n \leqq \frac{2}{3} M_n + 1$ which is a contradiction.

REMARK *added in proof.* As observed by L. Dor, if the estimate for $|S|$ is changed to the better one $|S| \leqq (ne/k)^k$ (for $k \leqq n/4$) then our proof of Theorem 2.1 will give the recent result of J. Elton $|A| \geqq cM_n^2/n \log(n/M_n)$ for suitable absolute constant c.

2.2. REMARK (due to G. Schechtman). If $\{x_i(t)\}_1^n$ is a set of characters on $T = G$ (as in 1), then the set S in the proof has the group property ($\vec{\varepsilon} = (\vec{\varepsilon}_i)_1^n \in S$ and $\vec{\delta} = (\delta_i)_1^n \in S$ imply $\vec{\varepsilon} \circ \vec{\delta} = (\varepsilon_i \delta_i)_{i=1}^n \in S$). In this case it is not necessary to use Lemma 1.2 and it is not hard to show that in case 2 in the proof, we have $|S| < 2^k$.† The same proof as before precisely gives in that case G. Pisier's result $|A| \geqq CM_n^2/n$ (one has to remember that the only case which needs to be considered is $M_n^2 \gg n \ln n$).

† We indicate a proof of this fact. If we consider E_2^n as n-dimensional space over the field $\{0; 1\}$ then S is a subspace. If $|S| \geqq 2^k$ then $a = \dim S \geqq k$. The standard linear algebraic argument (elementary row and column operations) shows that there exist $A \subset [1, \cdots, n]$, $|A| = a$, and a basis $\{e_j\}_{j=1}^a$ of S such that $\{e_j\}_{j=1}^a$ restricted on A is the canonical basis of E_2^A.

2.3. THEOREM. *Let $x_i(t)$ be real valued functions such that $|x_i(t)|=1$. Assume that* $\operatorname{span}\{x_i\}_1^n = X_n$ *cannot be isometrically embedded in l_∞^N. Then there exists $A \subset [1,\cdots,n]$ such that $k=|A|>\ln N/\ln n - 1$ and $\{x_i\}_{i\in A}$ is isometric to the natural basis of l_1^k.*

PROOF. Again, as in the proof of Theorem 2.1, we have for $k=\ln N/\ln n$ either case 1 (which gives the Theorem) or case 2. Now, in case 2, we consider the functions $\{y_{\bar\varepsilon}(t)=\chi_{T_{\bar\varepsilon}}(t)=$ the characteristic function of the set $T_{\bar\varepsilon}\}_{\bar\varepsilon\in S}$. The set $\{y_{\bar\varepsilon}(t)\}_{\bar\varepsilon\in S}$ gives (in C-norm) the natural basis in $l_\infty^{|S|}$. Therefore $X_n \hookrightarrow l_\infty^{|S|}$. So $|S|>N$. But it was shown before that $|S|<kn^k$. This gives $k>\log N/\log n - 1$.

2.4. COROLLARY. *Let $\{x_i(t)\}_1^n$ be as in 2. Let* $\operatorname{span}\{x_i\}_1^n = X$ *contain a 2-isomorphic copy of l_2^r. Then we can take in Theorem 2.3 $k \geqq Cr/\ln n$.*

The proof follows from the fact that if l_∞^N contains a 2-isomorphic copy of l_2^r then $N \geqq \exp cr$ for some absolute constant c ([7]).

2.5. COROLLARY. *Let $\{x_i(t)\}_1^n$ and $X_n=$* $\operatorname{span}\{x_i\}_1^n$ *be as in 2. Let X_n be an n-dimensional space with a cotype q constant K_q. Then we can take in Theorem 2.3*

$$k \geqq c(K_q,q)\frac{n}{(\ln(n+1))^{1+q}}\ .$$

In that case X_n contains a 2-isomorphic copy of l_2^r for $r \geqq cn^{2/q}$ by [2]. However, this observation gives only

$$k \geqq c\cdot K_q\frac{n^{2/q}}{\ln n}\ .$$

A more delicate argument uses an unpublished result of B. Maurey and gives (see G. Pisier [10]) that n-dimensional space X with a cotype q constant K_q cannot be embedded 2-isomorphically in l_∞^N for

$$\ln N < C(K_q,q)\frac{n}{(\ln(n+1))^{q}}$$

where $C(K_q)$ depends only on K_q and q. It is unknown (see [10]) if $(\ln n)^q$ is really necessary.

2.6. COROLLARY. *Let $X_n=$* $\operatorname{span}\{x_i(t)\}_1^n$ *be as in 2 and have dimension n. Assume that for no sequence $p(n)\to\infty$ can we find inside X_n for all large n a 2-isomorphic copy of $l_\infty^{p(n)}$. It follows from the Maurey–Pisier Theorem* [6] (see also [8]) *that X_n has a cotype q constant K_q (for some $q<\infty$) uniformly bounded independent of n. Then*

$$\operatorname*{Av}_{\varepsilon_i=\pm 1}\left\|\sum_1^n \varepsilon_i x_i\right\| = M_n \geqq cn/(\ln n)^\alpha$$

for some constant c and some number α. (Use Corollary 2.5.)

REMARK. If $\{x_i\}_{i=1}^n$ is a subset of characters as in 1.1 and $x_i(t)$ are real valued functions, then by Remark 2.2 we can reduce one $\ln n$ in all previous results 2.3–2.6.

3. In this section we will use more information about type and cotype. Let X be an n-dimensional normed space, let $\beta_m^{(q)} = \beta_m^{(q)}(X)$ be cotype q constants for m-vector subsets of X and let $\alpha_m^{(p)}$ be type p constants for m-vector subsets of X (see [6]). It is known (and easy [6]) that

$$\beta_m^{(q)}(X^*) \leqq \alpha_m^{(p)}(X) \qquad (1/q + 1/p = 1, X^* \text{ is a dual space to } X).$$

It is remarkable (and non-trivial) that for some absolute constant K,

$$\alpha_m^{(p)}(X) \leqq K(\ln n)\beta_m^{(q)}(X^*) \qquad \text{(G. Pisier, see [9])}.$$

3.1. The following known lemma is a simple consequence of the definition of $\beta_m^{(2)}$.

LEMMA. *If $\beta_m^{(2)}(X) \geqq cm^{1/2-1/q_0}$ for $m = m(n) \to \infty$ $(n \to \infty)$ then any cotype of E is at least q_0.*

3.2. THEOREM. *Let $\dim X_n = n$, $\alpha > 0$ and let X_n contain a 2-isomorphic copy of l_1^m for $m \geqq n^\alpha$. Then for any $\varepsilon > 0$ and any integer k there exists n_0 such that for $n \geqq n_0$ X_n contains a $(1+\varepsilon)$-isomorphic copy of l_1^k, which is $(1+\varepsilon)$-complemented.*

PROOF. It is clear that $\alpha_m^{(2)}(X) \geqq c\sqrt{m}$. It follows from 3 that

$$\beta_m^{(2)}(X_n^*) \geqq \frac{c}{K}\sqrt{m/\ln n}$$

and, by 3.1, X_n^* has no finite cotype. By the Maurey–Pisier theorem [6] for given $\varepsilon > 0$ and some $k = k(n) \to \infty$ $(n \to \infty)$ X_n^* contains a $(1+\varepsilon)$-isomorphic copy of l_∞^k and this implies the theorem.

3.3. THEOREM. *Let $\{x_i(t)\}_1^n$ and X_n be as in 2, and $\dim X_n = n$. Then for any $\varepsilon > 0$ and k there exists n_0 such that for $n \geqq n_0$ X_n contains either a $(1+\varepsilon)$-isomorphic copy of l_∞^k or X_n contains a $(1+\varepsilon)$-isomorphic and $(1+\varepsilon)$-complemented copy of l_1^k.*

PROOF. Follows from 2.5 and 3.2.

4. In this section we consider a more general view on the result of type 3.2. For studying the structure of finite-dimensional normed spaces it is very useful and promising to consider in some sense "large" subspaces. Theorem 3.2 gives such an example. To introduce a precise language for a notion of "large" we need a few definitions. Let

$$\mathrm{Log}^{(k)} m = \underbrace{\log(\cdots(\log m))}_{k \text{ times}}.$$

We say that $m = m(n)\ (\leqq n)$ is k-Log equivalent to n and we write $m \sim^{\mathrm{Log}^{(k)}} n$ if there exists some fixed $\alpha > 0$ such that $\mathrm{Log}^{(k)} m \geqq \alpha\, \mathrm{Log}^{(k)} n$. We agree also that $m \sim^{\mathrm{Log}^{(0)}} n$ means that $m \geqq \alpha n$ for some $\alpha > 0$ and all n. Let $\{r_i(t)\}_1^m$ be the Rademacher functions on $[0,1]$,

$$\mathrm{Rad}_m E = \left\{ y \in L_2(E) : y = \sum_1^n r_i(t) x_i,\ x_i \in E \right\} \subset L_2(E).$$

We say that p is a k-type (q is a k-cotype) of a family X of normed spaces $X = \{E_n, \dim E_n = n\}_{n\to\infty}$ iff $\forall \alpha > 0\ \exists T_p(\alpha)\ (C_q(\alpha))$ such that for any n and any $\{x_i \in E_n\}_i^m\ \|x_i\| = 1$, such that $\mathrm{Log}^{(k)} m \geqq \alpha\, \mathrm{Log}^{(k)} n$

$$\left(\frac{1}{C_q(\alpha)} m^{1/q} \leqq \right) \qquad \left\| \sum_1^m r_i(t) x_i \right\|_{L_2(E_n)} \leqq T_p(\alpha) m^{1/p}$$

(for 1-type we say sometimes a power-type and similarly a power-cotype). We will refer to the smallest function $T_p(\alpha)$ as the characteristic function of k-type p (or similarly for a cotype).

Let $p_k(X) = \sup\{p : p \text{ is } k\text{-type of } X\}$ and $q_k(X) = \inf\{q : q \text{ is } k \text{ cotype of } X\}$. In the same sense as was explained in 1, we use the notations $p_k(E_n)$ and $q_k(E_n)$ for a given n-dimensional space E_n but we mean a family X of spaces E_n which possess the same indicated properties. It is clear that $p_k(X)$ decreases and $q_k(X)$ increases when $k \nearrow$.

4.1. The following result is the main reason for introducing the notions of k-types and k-cotypes. Its proof is based on the result of G. Pisier [9] which we mentioned in 3. Let $K(n;m)$ be the norm of the canonical projection (called the Rademacher projection) $P_m : L_2(E_n) \to \mathrm{Rad}_m(E_n)$, $\dim E_n = n$. G. Pisier [9] has proved that $\|P_m\| \leqq K \ln(n+1)$ for some absolute constant K (it is easy to verify that $\alpha_m^{(p)}(X) \leqq \|P_m\| \beta_m^{(q)}(X^*)$ and it leads to the result mentioned in Section 3).

THEOREM (duality). *Let X and X^* be dual families of normed (finite-dimensional) spaces (i.e., $E_n \in X$ iff $E_n^* \in X^*$). Then for any $k \geqq 1$*

$$1/p_k(X) + 1/q_k(X^*) = 1.$$

PROOF. First, the standard simple argument gives that if p is a k-type for X then the dual number q (i.e. $1/p + 1/q = 1$) is a k-cotype for X^* (and even with the same function $T_p(\alpha)$ — see the definition). So it is enough to prove that X^* has k-cotype q implies that for any $\varepsilon > 0$, X has k-type $(p-\varepsilon)$. To show this we prove the following inequality.

LEMMA. *Let $T_p(\alpha; E)$ be a characteristic function of k-type p of the n-dimensional space E and let $C_q(\alpha; E^*)$ be the characteristic function of k-cotype q of E^* where $1/p + 1/q = 1$ and $k \geqq 1$. Then there exists some absolute constant C such that*

$$T_p(\alpha; E) \leqq C(\ln n)^2 \cdot C_q(\alpha/p; E^*).$$

REMARK. For $k > 1$ we can take $C_q(\alpha)$ instead of $C_q(\alpha/p)$.

PROOF. Let $X = \Sigma_1^m r_i(t)x_i \in L_2(E)$, $\|x_i\| = 1$, where $\mathrm{Log}^{(k)} m \geqq \alpha \,\mathrm{Log}^{(k)} n$. Let P_m be the Rademacher projection. It is sufficient to prove that

$$\|X\|_{L_2(E)} \leqq 6C_q\left(\frac{\alpha}{p}\right) \ln m \cdot \|P_m\| m^{1/p}.$$

There exists $F = \Sigma_1^m r_i(t) f_i \in \mathrm{Rad}_m(E^*)$ such that

$$\|X\|_{L_2(E)} \lesssim \|P_m\| \cdot \frac{(X, F)}{\|F\|_{L_2(E^*)}} \leqq \|P_m\| \frac{\sum_1^m \|f_i\|}{\|F\|_{L_2(E^*)}}.$$

Assume that $\max\|f_i\| = 1$. Take the following partition of $[1, \cdots, m]$:

$$A_j = \{i \in [1, \cdots, m] : 1/3^j \leqq \|f_i\| \leqq 1/3^{j-1}\}, \quad j = 1, \cdots.$$

Let $\mathrm{I} = \{j : |A_j| < 2^j m^{1/p}\}$ and $\mathrm{II} = \{j : |A_j| \geqq 2^j m^{1/p}\}$. We have

$$\sum_{j \in \mathrm{I}} \left(\sum_{i \in A_j} \|f_i\|\right) \leqq \sum_j 2^j m^{1/p} \cdot 1/3^{j-1} < 6m^{1/p}.$$

To estimate the sum over $j \in \mathrm{II}$ we remark that the cardinality of II cannot be too large:

$$m > \sum_{j \in \mathrm{II}} |A_j| \geqq \sum_{j \in \mathrm{II}} 2^j m^{1/p} > 2^{|\mathrm{II}|} m^{1/p}.$$

So $k = |\mathrm{II}| \leqq (1/q)\log m$. Inside one subset A_j we may use an inequality which follows from a cotype condition:

$$\sum_{i\in A_j} \|f_i\| \leqq |A_j| \cdot \frac{1}{3^{j-1}} = |A_j|^{1/p} \cdot |A_j|^{1/q} \frac{1}{3^{j-1}}$$

$$\leqq 3C^q\left(\frac{\alpha}{p}\right) \cdot |A_j|^{1/p}\left(\int \left\|\sum_{i\in A_j} r_i(t) f_i\right\|^2 dt\right)^{1/2} \leqq 3C^q\left(\frac{\alpha}{p}\right) m^{1/p}\|F\|_{L_2(E_n^*)}.$$

Finally we have

$$\sum_1^m \|f_i\| \leqq C^q\left(\frac{\alpha}{p}\right)\left(\frac{3\log m}{q}+6\right) m^{1/p} \cdot \|F\|_{L_2(E_n^*)}.$$

4.2. Corollary. *Let for some $k \geqq 0$, $p_k(E_n)=1$. Then for any $\varepsilon > 0$, E_n contains a $(1+\varepsilon)$-isomorphic and $(1+\varepsilon)$-complemented copy of l_1^t where $t = t(n;\varepsilon)\to\infty$ for $n\to\infty$ and fixed ε.*

It is clear that this fact is a generalization of Theorem 3.2 and has the same proof (use the Duality Theorem 4.1).

4.3. If X_n has a k-cotype q (for some $k \geqq 0$) then X_n contains a 2-isomorphic copy of l_2^r for $r \geqq cn^{2/q}$.

Proof is standard (see [2]) because it uses a cotype condition for λn vectors of an almost equal length.

4.4. It follows from the proof of theorem 3.3 from [1] that for any $k \geqq 2$ if $1 \leqq p_k < 2$ and p_k is k-type of a family $X = \{E_n\}$ then E_n (for n large enough) contains for any $\varepsilon > 0$ a $(1+\varepsilon)$-isometric copy of $l_{p_k}^m$ and m is k-Log equivalent to n.

4.5. It is known (see [4] and [14]) that for n-dimensional space E

$$d(E, l_2^n) \leqq 4\alpha_n^{(2)}(E)\cdot\beta_n^{(2)}(E) \leqq 4\alpha^{(p)}(E)\beta^{(q)}(E)n^{1/p-1/q}$$

where $\alpha^{(p)}$ is a type p constant and $\beta^{(q)}$ is a cotype q constant of E. We would like to have the same kind of formula for power type and cotype.

Statement. Let E be an n-dimensional normed space with a 1-type p (i.e. power-type p) characteristic function $T_p(\lambda)$ and a 1-cotype q characteristic function $C_q(\lambda)$. Let $1 \leqq p < 2$ and $q > 2$. Then for some absolute constant C

$$d(E, l_2^n) \leqq CT_p\left(\frac{1}{p}-\frac{1}{2}\right) C_q\left(\left(1-\frac{1}{q}\right)\left(\frac{1}{2}-\frac{1}{q}\right)\right)(\ln(n+1))^{3+1/q-1/p} n^{1/p-1/q}.$$

PROOF. We will obtain this formula by estimating $\alpha_n^{(2)}(E)$ and $\beta_n^{(2)}(E)$ from above.

LEMMA. $\alpha_n^{(2)}(E) \leqq 6[\ln(n+1)]^{1-1/p} T_p(1/p-1/2) n^{1/p-1/2}$.

PROOF. Similar to the proof of the lemma in 4.1. Let $\{x_i\}_1^n \subset E$ and $\|x_i\| \leqq 1$. Let $A_j = \{i : (1/3)^j \leqq \|x_i\| \leqq (1/3)^{j-1}\}$ and take $\alpha = 1/p - 1/2 > 0$. We have

$$(1) \quad \int \left\| \sum_1^n r_i(t) x_i \right\| dt \leqq \sum_j \int \left\| \sum_{i \in A_j} r_i(t) x_i \right\| dt \leqq \sum_j (1/3)^{j-1} \int \left\| \sum_{i \in A_j} r_i(t) x_i / \|x_i\| \right\|.$$

Let $\mathrm{I} = \{j : |A_j| < 2^j n^\alpha\}$ and $\mathrm{II} = \{j : |A_j| \geqq 2^j n^\alpha\}$. It is clear that (see 4.1, lemma)

$$|\mathrm{II}| \leqq \frac{\ln n}{(1-\alpha)\ln 2} \leqq 2 \ln n.$$

We continue now the inequality (1) using the triangle inequality for $j \in \mathrm{I}$ and the power type condition for $j \in \mathrm{II}$:

$$\int \left\| \sum_1^n r_i(t) x_i \right\| dt \leqq \sum_{j \in \mathrm{I}} \left(\frac{1}{3}\right)^{j-1} 2^j n^\alpha + \sum_{j \in \mathrm{II}} \left(\frac{1}{3}\right)^{j-1} T_p(\alpha) \cdot |A_j|^{1/p}$$

$$\leqq 3n^\alpha \sum_j \left(\frac{2}{3}\right)^j + 3T_p(\alpha) \sum_{j \in \mathrm{II}} \left(\frac{1}{3}\right)^j |A_j|^{1/p}$$

$$\leqq Cn^\alpha + 3T_p(\alpha) \left(\sum_{j \in \mathrm{II}} \left(\frac{1}{3}\right)^{2j} |A_j| \right)^{1/2} \left(\sum_{j \in \mathrm{II}} |A_j|^{(2-p)/p} \right)^{1/2}.$$

On the other hand

$$\left(\sum \|x_i\|^2 \right) \geqq \sum_j \sum_{i \in A_j} \left(\frac{1}{3}\right)^{2j} \geqq \sum_{j \in \mathrm{II}} |A_j| \left(\frac{1}{3}\right)^{2j}$$

and by Hölder's inequality for $p_1 = p/(2-p)$ and $q_1 = p/2(p-1)$

$$\sum_{j \in \mathrm{II}} |A_j|^{(2-p)/p} \leqq \left(\sum_{j \in \mathrm{II}} |A_j| \right)^{(2-p)/p} \cdot |\mathrm{II}|^{2(p-1)/p} \leqq n^{2(1/p-1/2)} (2 \ln n)^{2(1-1/p)}.$$

Therefore

$$\int \left\| \sum_1^n r_i(t) x_i \right\| dt \leqq 6n^\alpha + 6T(\alpha)(\ln n)^{1-1/p} n^{1/p-1/2} \left(\sum_1^n \|x_i\|^2 \right)^{1/2}$$

and this proves the lemma.

To estimate $\beta_n^{(2)}(E)$ we will pass to E^*. It is well known (see [6] or [2]) that $\beta_n^{(2)}(E) \leqq \alpha_n^{(2)}(E^*)$. So, by the previous lemma

$$\beta_n^{(2)}(E) \leqq 6(\ln(n+1))^{1/q} T_{q'}(1/2 - 1/q; E^*) n^{1/q' \; 1/2}$$

where $1/q' + 1/q = 1$ and $T_{q'}$ is a 1-type q' characteristic function for E^*. We have, by the lemma from Section 4.1,

$$\beta_n^{(2)}(E) \leqq C \cdot (\ln(n+1))^{1/q+2} C_q((1/2 - 1/q) \cdot 1/q'; E) n^{1/2-1/q},$$

for some absolute constant C. Finally, we obtain the formula using inequality $d(E, l_2^n) \leqq 4\alpha_n^{(2)}(E) \cdot \beta_n^{(2)}(E)$ which was mentioned at the beginning of this section.

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